

Motivic decompositions of projective homogeneous varieties and change of coefficients

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Abstract

We prove that under some assumptions on an algebraic group G , indecomposable direct summands of the motive of a projective G -homogeneous variety with coefficients in \mathbb{F}_p remain indecomposable if the ring of coefficients is any field of characteristic p . In particular for any projective G -homogeneous variety X , the decomposition of the motive of X in a direct sum of indecomposable motives with coefficients in any finite field of characteristic p corresponds to the decomposition of the motive of X with coefficients in \mathbb{F}_p . We also construct a counterexample to this result in the case where G is arbitrary.

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Résumé

Décompositions motiviques des variétés projectives homogènes et changement des coefficients.

Nous prouvons que sous certaines hypothèses sur un groupe algébrique G , tout facteur direct indécomposable du motif associé à une variété projective G -homogène à coefficients dans \mathbb{F}_p demeure indécomposable si l'anneau des coefficients est un corps de caractéristique p . En particulier pour toute variété projective G -homogène X , la décomposition du motif de X comme somme directe de motifs indécomposables à coefficients dans tout corps fini de caractéristique p correspond à la décomposition du motif de X à coefficients dans \mathbb{F}_p . Nous exhibons de plus un contre-exemple à ce résultat dans le cas où le groupe G est quelconque.

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Introduction

Let F be a field, Λ be a commutative ring, $CM(F; \Lambda)$ be the category of *Grothendieck Chow motives* with coefficients in Λ , G a semi-simple affine algebraic group and X a projective G -homogeneous F -

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variety. The purpose of this note is to study the behaviour of the complete motivic decomposition (in a direct sum of indecomposable motives) of $X \in CM(F; \Lambda)$ when changing the ring of coefficients. In the first part we prove some very elementary results in non-commutative algebra and find sufficient conditions for the tensor product of two connected rings to be connected. In the second part we show that under some assumptions on G , indecomposable direct summands of X in $CM(F; \mathbb{F}_p)$ remain indecomposable if the ring of coefficients is any field of characteristic p (Theorem 2.1), since these conditions hold for the *reduced endomorphism ring* of indecomposable direct summands of X . In particular theorem 2.1 implies that the complete decomposition of the motive of X with coefficients in any finite field of characteristic p corresponds to the complete decomposition of the motive of X with coefficients in \mathbb{F}_p . Finally we show that theorem 2.1 doesn't hold for arbitrary G by producing a counterexample.

Let Λ be a commutative ring. Given a field F , an F -variety will be understood as a separated scheme of finite type over F . Given such Λ and an F -variety X , we can consider $CH_i(X; \Lambda)$, the Chow group of i -dimensional cycles on X modulo rational equivalence with coefficients in Λ , defined as $CH_i(X) \otimes_{\mathbb{Z}} \Lambda$. These groups are the first step in the construction of the category $CM(F; \Lambda)$ of *Grothendieck Chow motives* with coefficients in Λ . This category is constructed as the *pseudo-abelian envelope* of the category $CR(F; \Lambda)$ of *correspondences* with coefficients in Λ . Our main reference for the construction and the main properties of these categories is [2]. For a field extension E/F and any correspondence $\alpha \in CH(X \times Y; \Lambda)$ we denote by α_E the pull-back of α along the natural morphism $(X \times Y)_E \rightarrow X \times Y$. Considering a morphism of commutative rings $\varphi : \Lambda \rightarrow \Lambda'$ we define the two following functors. The *change of base field* functor is the additive functor $res_{E/F} : CM(F; \Lambda) \rightarrow CM(E; \Lambda)$ which maps any summand $(X, \pi)[i] \in CM(F; \Lambda)$ to $(X_E, \pi_E)[i]$ and any morphism $\alpha : (X, \pi)[i] \rightarrow (Y, \rho)[j]$ to α_E . The *change of coefficients* functor is the additive functor $coef_{\Lambda'/\Lambda} : CM(F; \Lambda) \rightarrow CM(F; \Lambda')$ which maps any summand $(X, \pi)[i]$ to $(X, (id \otimes \varphi)(\pi))[i]$ and any morphism $\alpha : (X, \pi)[i] \rightarrow (Y, \rho)[j]$ to $(id \otimes \varphi)(\alpha)$.

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1. On the tensor product of connected rings

Recall that a ring A is *connected* if there are no idempotents in A besides 0 and 1.

Proposition 1.1 *Let A be a finite and connected ring. Then any element a in A is either nilpotent or invertible. The set \mathcal{N} of nilpotent elements in A is a two-sided and nilpotent ideal.*

In order to prove Proposition 1.1 we will need the following elementary lemma.

Lemma 1.2 *Let A be a finite ring. An appropriate power of any element a of A is idempotent.*

Proof. For any $a \in A$, the set $\{a^n, n \in \mathbb{N}\}$ is finite, hence there is a couple $(p, k) \in \mathbb{N}^2$ (with k non-zero) such that $a^p = a^{p+k}$. The sequence $(a^n)_{n \geq p}$ is k -periodic and for example if s is the lowest integer such that $p < sk$, a^{sk} is idempotent.

Proof of Proposition 1.1. For any $a \in A$, an appropriate power of a is an idempotent by lemma 1.2. Since A is connected, this power is either 0 or 1, that is to say a is either nilpotent or invertible.

We now show that the set \mathcal{N} of nilpotent elements in A is a two-sided ideal. First if a is nilpotent in A , then for any b in A , ab and ba are not invertible, hence ab and ba belong to \mathcal{N} .

It remains to show that the sum of two nilpotent elements in A is nilpotent. Setting ν for the number of nilpotent elements in A , we claim that for any sequence a_1, \dots, a_ν in \mathcal{N} , $a_1 \dots a_\nu = 0$. Indeed if $a_{\nu+1}$ is any nilpotent in A the finite sequence $\Pi_1 = a_1, \Pi_2 = a_1 a_2, \dots, \Pi_{\nu+1} = a_1 a_2 \dots a_{\nu+1}$ consists of nilpotents and by the pigeon-hole principle $\Pi_k = \Pi_s$, for some k and s satisfying $1 \leq k < s \leq \nu + 1$. Therefore $\Pi_s = \Pi_k a_{k+1} \dots a_s = \Pi_k$ which implies that $\Pi_k(1 - a_{k+1} \dots a_s) = 0$ and $\Pi_k = 0$ since $1 - a_{k+1} \dots a_s$ is

invertible. With this in hand it is clear that for any a and b in \mathcal{N} , $(a+b)^\nu = 0$. Furthermore $\mathcal{N}^\nu = 0$ and \mathcal{N} is nilpotent.

Corollary 1.3 *Let A be a finite and connected \mathbb{F}_p -algebra endowed with a ring morphism $\varphi : A \rightarrow \mathbb{F}_p$. Then the set \mathcal{N} of nilpotent elements in A is precisely $\ker(\varphi)$. Furthermore for any connected \mathbb{F}_p -algebra E , $A \otimes_{\mathbb{F}_p} E$ is connected.*

Proof. For any $a \in \mathcal{N}$ and $n \in \mathbb{N}^*$ such that $a^n = 0$, $0 = \varphi(a^n) = \varphi(a)^n$, hence a lies in the kernel of φ . On the other hand if $\varphi(a) = 0$, a is not invertible thus a is nilpotent and $\mathcal{N} = \ker(\varphi)$. Since \mathcal{N} is nilpotent, $\mathcal{N} \otimes E$ is also nilpotent. The sequence

$$0 \longrightarrow \mathcal{N} \otimes E \longrightarrow A \otimes E \xrightarrow{\psi} E \longrightarrow 0$$

is exact and we want to show that any idempotent P in $A \otimes_{\mathbb{F}_p} E$ is either 0 or 1. Since E is connected, $\psi(P)$ is either 0 or 1. We may replace P by $1 - P$ and so assume that P lies in the kernel of ψ , which implies that the idempotent P is nilpotent, hence $P = 0$.

2. Application to motivic decompositions of projective homogeneous varieties

For any semi-simple affine algebraic group G , the full subcategory of $CM(F; \Lambda)$ whose objects are finite direct sums of twists of direct summands of the motives of projective G -homogeneous F -varieties will be denoted $CM_G(F; \Lambda)$. We now use corollary 1.3 to study how motivic decompositions in $CM_G(F; \Lambda)$ behave when extending the ring of coefficients. A pseudo-abelian category \mathcal{C} satisfies the *Krull-Schmidt principle* if the monoid (\mathfrak{C}, \oplus) is free, where \mathfrak{C} denotes the set of the isomorphism classes of objects of \mathcal{C} .

In the sequel Λ will be a connected ring and X an F -variety. A field extension E/F is a *splitting field* of X if the E -motive X_E is isomorphic to a finite direct sum of twists of Tate motives. The F -variety X is *geometrically split* if X splits over an extension of F , and X satisfies the *nilpotence principle*, if for any field extension E/F the kernel of the morphism $res_{E/F} : End(M(X)) \rightarrow End(M(X_E))$ consists of nilpotents. Any projective homogeneous variety (under the action of a semi-simple affine algebraic group) is geometrically split and satisfies the nilpotence principle (see [1]), therefore if Λ is finite the Krull-Schmidt principle holds for $CM_G(F; \Lambda)$ by [5, Corollary 3.6], and we can serenely deal with motivic decompositions in $CM_G(F; \Lambda)$.

Let G be a semi-simple affine adjoint algebraic group over F and p a prime. The absolute Galois group $Gal(F_{sep}/F)$ acts on the Dynkin diagram of G and we say that G is of *inner type* if this action is trivial. By [1] the subfield F_G of F_{sep} corresponding to the kernel of this action is a finite Galois extension of F , and we will say that G is *p-inner* if $[F_G : F]$ is a power of p . We now state the main result.

Theorem 2.1 *Let G be a semi-simple affine adjoint p -inner algebraic group and $M \in CM_G(F; \mathbb{F}_p)$. Then for any field L of characteristic p , M is indecomposable if and only if $coeff_{L/\mathbb{F}_p}(M)$ is indecomposable.*

If X is geometrically split the image of any correspondence $\alpha \in CH_{\dim(X)}(X \times X; \Lambda)$ by the *change of base field* functor $res_{E/F}$ to a splitting field E/F of X will be denoted $\overline{\alpha}$. The *reduced endomorphism ring* of any direct summand (X, π) is defined as $res_{E/F}(End_{CM(F; \Lambda)}((X, \pi)))$ and denoted by $\overline{End}((X, \pi))$.

Let X be a complete and irreducible F -variety. The pull-back of the natural morphism $Spec(F(X)) \times X \rightarrow X \times X$ gives rise to $mult : CH_{\dim(X)}(X \times X; \Lambda) \rightarrow CH_0(X_{F(X)}; \Lambda) \rightarrow \Lambda$ (where the second map is the usual *degree* morphism). For any correspondence $\alpha \in CH_{\dim(X)}(X \times X; \Lambda)$, $mult(\alpha)$ is called the *multiplicity* of α and we say that a direct summand (X, π) given by a projector $\pi \in CH_{\dim(X)}(X \times X; \Lambda)$ is *upper* if $mult(\pi) = 1$. If (X, π) is an upper direct summand of a complete and irreducible F -variety, the multiplicity $mult : End_{CM(F; \Lambda)}((X, \pi)) \rightarrow \Lambda$ is a morphism of rings by [4, Corollary 1.7].

Proposition 2.1 *Let G be a semi-simple affine algebraic group and $M = (X, \pi) \in CM(F; \mathbb{F}_p)$ the upper direct summand of the motive of an irreducible and projective G -homogeneous F -variety. Then for any*

field L of characteristic p , M is indecomposable if and only if $\text{coeff}_{L/\mathbb{F}_p}(M)$ is indecomposable.

Proof. Since the change of coefficients functor is additive and maps any non-zero projector to a non-zero projector, it is clear that if $\text{coeff}_{L/\mathbb{F}_p}(M)$ is indecomposable, M is also indecomposable. Considering a splitting field E of X , the reduced endomorphism ring $\overline{\text{End}}(M) := \overline{\pi} \circ \overline{\text{End}}(X) \circ \overline{\pi}$ is connected since M is indecomposable and finite. Corollary 1.3, with $A = \overline{\text{End}}(M)$, $E = L$ and $\varphi = \text{mult}$ implies that $\overline{\text{End}}(M) \otimes L = \overline{\text{End}}(\text{coeff}_{L/\mathbb{F}_p}(M))$ is connected, therefore by the nilpotence principle $\text{End}(\text{coeff}_{L/\mathbb{F}_p}(M))$ is also connected, that is to say $\text{coeff}_{L/\mathbb{F}_p}(M)$ is indecomposable.

Proof of Theorem 2.1. Recall that G is a semi-simple affine adjoint p -inner algebraic group and consider a projective G -homogeneous F -variety X . By [6, Theorem 1.1], any indecomposable direct summand M of X is a twist of the upper summand of the motive of an irreducible and projective G -homogeneous F -variety Y , thus we can apply proposition 2.1 to each indecomposable direct summand of X .

Remark : If Λ is a finite, commutative and connected ring, complete motivic decompositions in $CM(F; \Lambda)$ remain complete when the coefficients are extended to the residue field of Λ by [7, Corollary 2.6], hence the study of motivic decompositions in $CM_G(F; \Lambda)$, where Λ is any finite connected ring whose residue field is of characteristic p , is reduced to the study motivic decompositions in $CM_G(F; \mathbb{F}_p)$.

We now produce a counterexample to Theorem 2.1 in the case where the algebraic group G doesn't satisfy the needed assumptions. Let L/F be a Galois extension of degree 3. By [1, Section 7], the endomorphism ring $\text{End}(M(\text{Spec}(L)))$ of the motive associated with the F -variety $\text{Spec}(L)$ with coefficients in \mathbb{F}_2 is the \mathbb{F}_2 -algebra of $\text{Gal}(L/F)$, i.e. $\frac{\mathbb{F}_2[X]}{(X^3-1)} \simeq \mathbb{F}_2 \times \mathbb{F}_4$, hence $M(\text{Spec}(L)) = M \oplus N$, with $\text{End}(N) = \mathbb{F}_4$ and both M and N are indecomposable. Now $\text{End}(\text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)) = \mathbb{F}_4 \otimes \mathbb{F}_4$ is not connected since $1 \otimes \alpha + \alpha \otimes 1$ is a non-trivial idempotent for any $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$, hence $\text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)$ is decomposable.

Consider the $(PGL_2)_L$ -homogeneous L -variety \mathbb{P}_L^1 . The Weil restriction $\mathcal{R}(\mathbb{P}_L^1)$ is a projective homogeneous F -variety under the action of the Weil restriction of $(PGL_2)_L$, and the minimal extension such that $\mathcal{R}((PGL_2)_L)$ is of inner type is L . By [3, Example 4.8], the motive with coefficients in \mathbb{F}_2 of $\mathcal{R}(\mathbb{P}_L^1)$ contains two twists of $\text{Spec}(L)$ as direct summands, therefore at least two indecomposable direct summands of $\mathcal{R}(\mathbb{P}_L^1)$ split off over \mathbb{F}_4 .

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